

Institut des Mathématiques pour la Planète Terre

Crues, inondations, submersion Part 3

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Coastal flooding hazard

Setup: increase in mean water level

- Barometric setup
- Wind setup
- Wave setup

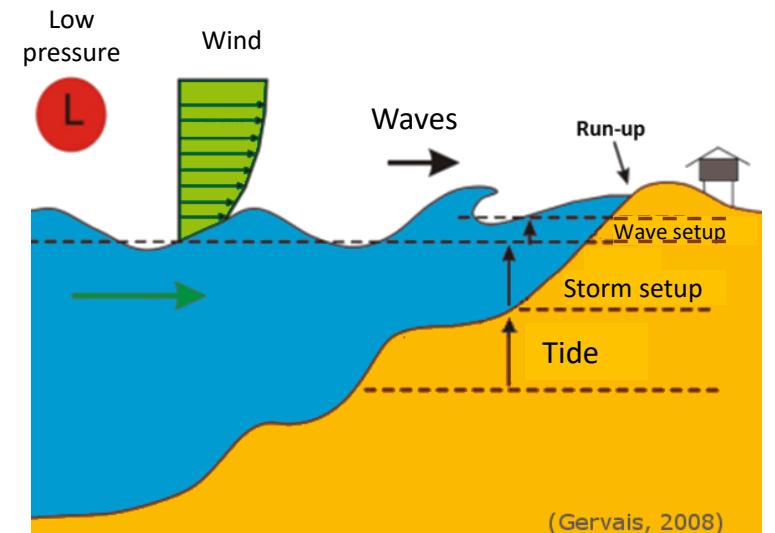
Setup forecasting

Exemple: partnership project SHOM-Météo France

HOMONIM project Better anticipate and manage coastal flooding hazard

Improve knowledge of coastal flooding hazard

→ Develop an operational modelling capacity of setup and waves



Unstructured Grids

Mesh used by SHOM at the regional scale
(courtesy of Rémy Baraille / SHOM)

2 600 000 cells



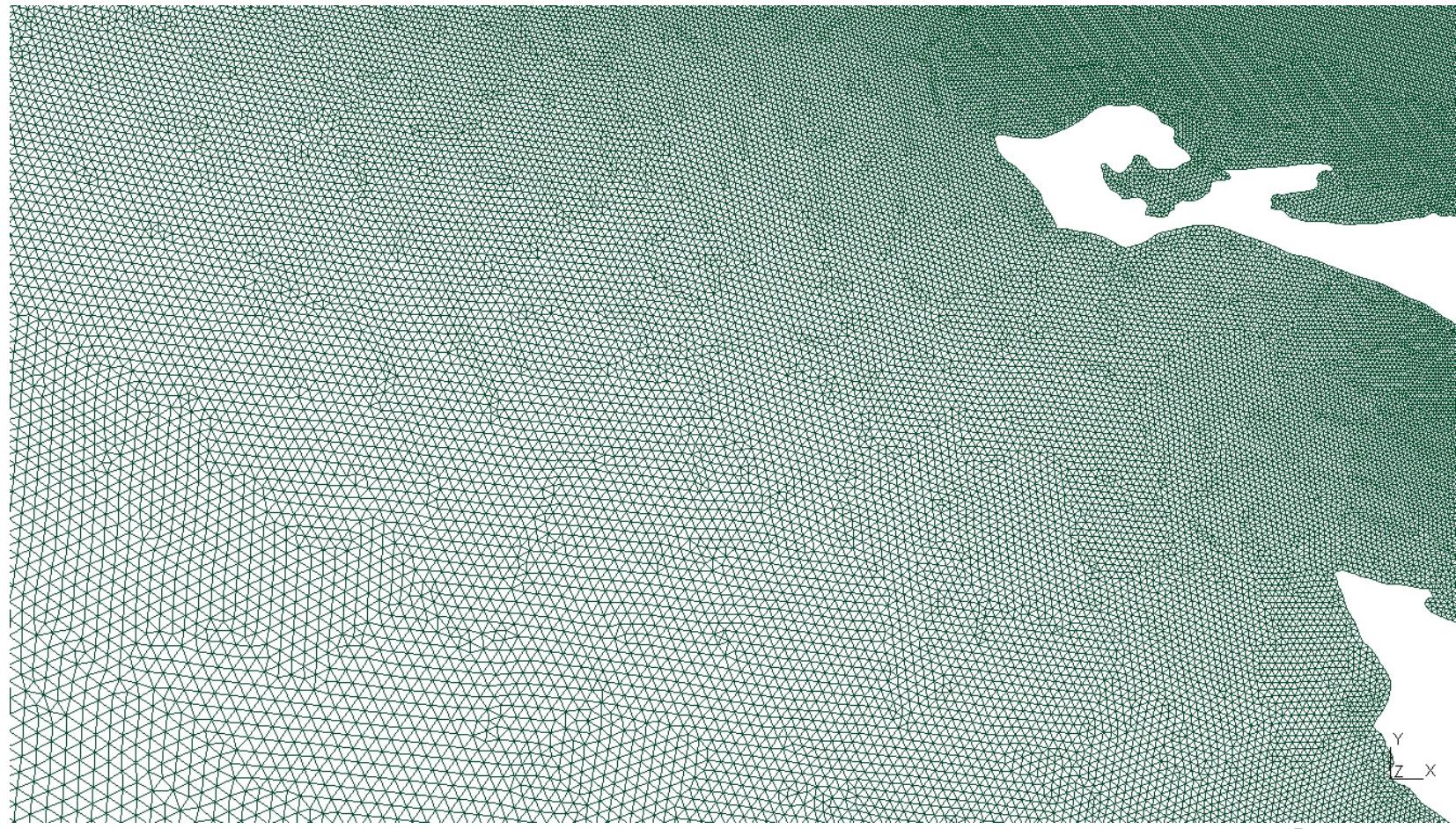
Unstructured Grids

Atlantic coast



Unstructured Grids

Île de Ré and Oléron island



Unstructured Grids

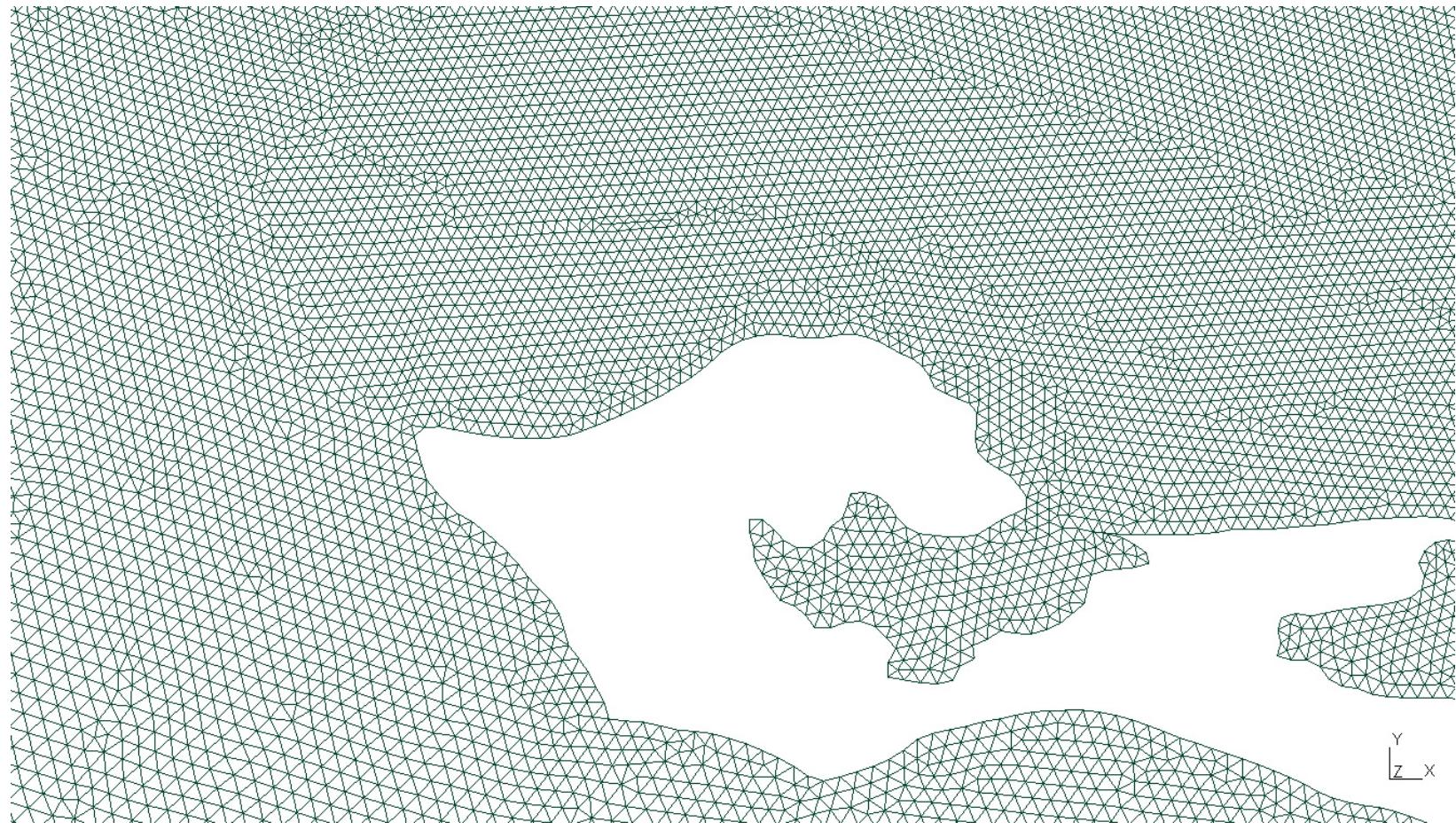
Île de Ré

Unstructured Meshes (triangular)

- Follow the shape of the coast
- Smaller cells where precision is needed

Waves reflection

Total number of cells



Computational Time

Coastal flooding forecast (Météo France / SHOM)

Five days forecast updated every hour

→ Computational time : 10 minutes

→ The calculation must be really **very fast** !

and accurate...



- Model → **2D depth-averaged models**
 - Atlantic regional configuration 200 m (HR) : shallow water model
 - Nord-Aquitaine coast configuration 50 m (THR) : shallow water model + phased-averaged wave model
 - Coastal dynamics (ex. Île de Ré) < 10 m (XTHR) : phase-resolved wave models including non-hydrostatic effects
- Numerical Scheme
- Code optimization
- Nesting
ex. SHOM
- Unstructured Meshes
- Parallelization

Derivation of the depth-averaged models

Equations of fluid mechanics

Mass conservation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

incompressibility $\operatorname{div} \mathbf{v} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\mathbf{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Momentum balance equation

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \rho \mathbf{g} - \operatorname{grad} p + \operatorname{div} \boldsymbol{\tau}$$

$$\begin{cases} \rho \left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} \right) = \rho g \sin \theta \cos \alpha - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\ \rho \left(\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} \right) = \rho g \sin \theta \sin \alpha - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\ \rho \left(\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} \right) = -\rho g \cos \theta - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{cases}$$

Boundary conditions

$$z = 0 \quad w(0) = 0 \quad \text{No-penetration}$$

$$u(0) = v(0) = 0 \quad \text{No-slip}$$

$$z = h$$

Kinematic condition

$$w(h) = \frac{\partial h}{\partial t} + u(h) \frac{\partial h}{\partial x} + v(h) \frac{\partial h}{\partial y}$$

Dynamic condition

$$\begin{cases} \left[p(h) - \tau_{xx}(h) \right] \frac{\partial h}{\partial x} - \tau_{xy}(h) \frac{\partial h}{\partial y} + \tau_{xz}(h) = 0 \\ \left[p(h) - \tau_{yy}(h) \right] \frac{\partial h}{\partial y} - \tau_{xy}(h) \frac{\partial h}{\partial x} + \tau_{yz}(h) = 0 \\ -\tau_{xz}(h) \frac{\partial h}{\partial x} - \tau_{yz}(h) \frac{\partial h}{\partial y} - p(h) + \tau_{zz}(h) = 0 \end{cases}$$

$\theta = 0$

Case of an incompressible perfect fluid on a horizontal bottom

Shallow water

$$\varepsilon = \frac{h_0}{L} \ll 1$$

Scaling

$$x' = \frac{x}{L} \quad y' = \frac{y}{L} \quad z' = \frac{z}{h_0} \quad h' = \frac{h}{h_0} \quad u' = \frac{u}{u_0} \quad v' = \frac{v}{u_0} \quad w' = \frac{w}{\varepsilon u_0} \quad t' = t \frac{u_0}{L} \quad p' = \frac{p}{\rho g h_0}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0$$

$$\begin{cases} \rho \left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} \right) = -\frac{\partial p}{\partial x} \\ \rho \left(\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} \right) = -\frac{\partial p}{\partial y} \\ \rho \left(\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} \right) = -\rho g - \frac{\partial p}{\partial z} \end{cases}$$

Hydrostatic pressure : $\mathbf{grad} p_H = \rho g$

$$\frac{\partial p_H}{\partial z} = -\rho g$$

$$p = p_{\text{atm}} + \rho g(h - z)$$

Boundary conditions

$$w'(0) = 0$$

$$w'(h) = \frac{\partial h'}{\partial t'} + u'(h) \frac{\partial h'}{\partial x'} + v'(h) \frac{\partial h'}{\partial y'}$$

$$p'(h) = p_{\text{atm}} = 0$$

$$\begin{cases} \frac{\partial u'}{\partial t'} + \frac{\partial u'^2}{\partial x'} + \frac{\partial u'v'}{\partial y'} + \frac{\partial u'w'}{\partial z'} = -\frac{1}{F^2} \frac{\partial p'}{\partial x'} \\ \frac{\partial v'}{\partial t'} + \frac{\partial u'v'}{\partial x'} + \frac{\partial v'^2}{\partial y'} + \frac{\partial v'w'}{\partial z'} = -\frac{1}{F^2} \frac{\partial p'}{\partial y'} \\ \varepsilon^2 \left(\frac{\partial w'}{\partial t'} + \frac{\partial u'w'}{\partial x'} + \frac{\partial v'w'}{\partial y'} + \frac{\partial w'^2}{\partial z'} \right) = -\frac{1}{F^2} \left(1 + \frac{\partial p'}{\partial z'} \right) \end{cases}$$

Froude number
(Bélanger, Reech)

$$F = \frac{u_0}{\sqrt{gh_0}} \quad F = O(1)$$

$$\begin{cases} \frac{\partial u'}{\partial t'} + \frac{\partial u'^2}{\partial x'} + \frac{\partial u'v'}{\partial y'} + \frac{\partial u'w'}{\partial z'} = -\frac{1}{F^2} \frac{\partial p'}{\partial x'} \\ \frac{\partial v'}{\partial t'} + \frac{\partial u'v'}{\partial x'} + \frac{\partial v'^2}{\partial y'} + \frac{\partial v'w'}{\partial z'} = -\frac{1}{F^2} \frac{\partial p'}{\partial y'} \\ 1 + \frac{\partial p'}{\partial z'} = O(\varepsilon^2) \end{cases}$$

“Hydrostatic” pressure \longrightarrow No dispersion

Depth-averaging

$$\langle A \rangle = \frac{1}{h} \int_0^h A dz \quad U = \langle u \rangle \quad V = \langle v \rangle$$

Mass

$$\int_0^{h'} \frac{\partial u'}{\partial x'} dz' + \int_0^{h'} \frac{\partial v'}{\partial y'} dz' + \int_0^{h'} \frac{\partial w'}{\partial z'} dz' = 0$$

$$\frac{\partial}{\partial x'} \int_0^{h'} u' dz' - u'(h) \frac{\partial h'}{\partial x'} + \frac{\partial}{\partial y'} \int_0^{h'} v' dz' - v'(h) \frac{\partial h'}{\partial y'} + w'(h) - w'(0) = 0$$

$$\frac{\partial h'}{\partial t'} + \frac{\partial h' U'}{\partial x'} + \frac{\partial h' V'}{\partial y'} = 0$$

The boundary conditions are “incorporated” in the depth-averaged equations.

Momentum

$$\frac{\partial p'}{\partial z'} = -1 + O(\varepsilon^2) \quad p' = h' - z' + O(\varepsilon^2) \quad \int_0^{h'} p' dz' = \frac{h'^2}{2} + O(\varepsilon^2)$$

$$\int_0^{h'} \frac{\partial u'}{\partial t'} dz' + \int_0^{h'} \frac{\partial u'^2}{\partial x'} dz' + \int_0^{h'} \frac{\partial u' v'}{\partial y'} dz' + \int_0^{h'} \frac{\partial u' w'}{\partial z'} dz' + \frac{1}{F^2} \int_0^{h'} \frac{\partial p'}{\partial x'} dz' = 0$$

Boundary conditions

$$w'(0) = 0$$

$$w'(h) = \frac{\partial h'}{\partial t'} + u'(h) \frac{\partial h'}{\partial x'} + v'(h) \frac{\partial h'}{\partial y'}$$

$$\frac{\partial}{\partial t'} \int_0^{h'} u' dz' - u'(h') \frac{\partial h'}{\partial t'} + \frac{\partial}{\partial x'} \int_0^{h'} u'^2 dz' - u'^2(h') \frac{\partial h'}{\partial x'} + \frac{\partial}{\partial y'} \int_0^{h'} u' v' dz' - u'(h') v'(h') \frac{\partial h'}{\partial y'} + u'(h') w'(h') - u'(0) w'(0) + \frac{1}{F^2} \frac{\partial}{\partial x'} \int_0^{h'} p' dz' = 0$$

$$\frac{\partial h' U'}{\partial t'} + \frac{\partial}{\partial x'} \left(h' \langle u'^2 \rangle + \frac{h'^2}{2F^2} \right) + \frac{\partial}{\partial y'} \left(h' \langle u' v' \rangle \right) = O(\varepsilon^2)$$

$$\frac{\partial h' V'}{\partial t'} + \frac{\partial}{\partial x'} \left(h' \langle u' v' \rangle \right) + \frac{\partial}{\partial y'} \left(h' \langle v'^2 \rangle + \frac{h'^2}{2F^2} \right) = O(\varepsilon^2)$$

Advantages of the depth-averaged equations

Advantages

- The dimension of the system is reduced by 1

2D flow \longrightarrow 1D Model

3D flow \longrightarrow 2D Model

- The boundary conditions are incorporated in the model equations

No need to follow the free surface, to calculate at each time step the calculation domain.

\longrightarrow Computational time saving and simplicity of the numerical scheme

Disadvantages

- Depth-averaged quantities, The information in the Oz direction is lost Not always !

Problems

Non-hydrostatic effects and dispersion

$$\varepsilon^2 \left(\frac{\partial w'}{\partial t'} + \frac{\partial u'w'}{\partial x'} + \frac{\partial v'w'}{\partial y'} + \frac{\partial w'^2}{\partial z'} \right) = -\frac{1}{F^2} \left(1 + \frac{\partial p'}{\partial z'} \right)$$

Shearing

$$\langle u^2 \rangle ? \quad \langle v^2 \rangle ? \quad \langle uv \rangle ?$$

Equations of Saint-Venant :

u and v are almost uniform over the depth.

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(h \langle u^2 \rangle + \frac{gh^2}{2} \right) + \frac{\partial}{\partial y} (h \langle uv \rangle) = 0$$

$$\frac{\partial hV}{\partial t} + \frac{\partial}{\partial x} (h \langle uv \rangle) + \frac{\partial}{\partial y} \left(h \langle v^2 \rangle + \frac{gh^2}{2} \right) = 0$$

$$\longrightarrow \quad \langle u^2 \rangle = U^2 \quad \langle v^2 \rangle = V^2 \quad \langle uv \rangle = UV$$

The shearing is neglected.

If the shearing is not negligible ?

Friction

Empirical laws (Chézy, Gauckler-Manning-Strickler, etc.)

Value of the friction coefficient ?

Reconstruction of the 3D structure of the velocity field ?

How can we keep the information in the Oz direction ?

Non-hydrostatic pressure

Important for coastal waves in the shoaling zone (before breaking)

Incompressible 2D case

Hydrostatic pressure + Non-hydrostatic pressure

$$p = p_H + p_N$$

$$\begin{cases} \frac{\partial p_H}{\partial z} = -\rho g \\ \frac{\partial p_N}{\partial z} = -\left(\frac{\partial \rho w}{\partial t} + \frac{\partial \rho uw}{\partial x} + \frac{\partial \rho w^2}{\partial z} \right) \end{cases}$$

Mass conservation + no-penetration

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ w(0) = 0 \end{cases} \longrightarrow w = - \int_0^z \frac{\partial u}{\partial x} dz$$

Scaling $p'_N = \frac{p_N}{\varepsilon^2 \rho g h_0}$

$$\rho \left(\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} \right) = -\rho g - \frac{\partial p}{\partial z}$$

$$\varepsilon^2 \left(\frac{\partial w'}{\partial t'} + \frac{\partial u'w'}{\partial x'} + \frac{\partial w'^2}{\partial z'} \right) = -\frac{1}{F^2} \left(1 + \frac{\partial p'}{\partial z'} \right)$$

Irrational case

$$u = U + u^*$$

$$\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial u'^*}{\partial z'} = \varepsilon^2 \frac{\partial w'}{\partial x'}$$

$$u' = U' + O(\varepsilon^2)$$

$$w' = -z' \frac{\partial U'}{\partial x'} + O(\varepsilon^2)$$

$$\dot{h} = \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} = -h \frac{\partial U}{\partial x}$$

$$\ddot{h} = \frac{\partial \dot{h}}{\partial t} + U \frac{\partial \dot{h}}{\partial x} \quad \ddot{h} = h \left[\left(\frac{\partial U}{\partial x} \right)^2 - \frac{\partial^2 U}{\partial x \partial t} - U \frac{\partial^2 U}{\partial x^2} \right]$$

Serre-Green-Naghdi equations

$$\frac{\partial}{\partial x} \int_0^h p_N dz = \frac{\partial}{\partial x} \left(\frac{h^2 \ddot{h}}{3} \right)$$

$$\frac{\partial h}{\partial t} + \frac{\partial h U}{\partial x} = 0$$

Serre (1953)
Green & naghdi (1974)

$$\frac{\partial h U}{\partial t} + \frac{\partial}{\partial x} \left(h U^2 + \frac{gh^2}{2} + \frac{h^2 \ddot{h}}{3} \right) = 0$$

Third derivative

Fully nonlinear : no assumption on the amplitude

Dispersion relation

$$h = h_0 + h'$$

$$U = U'$$

$$[h', U']^T = [A_1, A_2]^T e^{i(kx - \omega t)}$$

$$\omega^2 = gh_0 k^2 \frac{1}{1 + \frac{k^2 h_0^2}{3}}$$

Phase velocity

$$v_\varphi = \frac{\omega}{k} = \sqrt{gh_0} \sqrt{\frac{1}{1 + \frac{k^2 h_0^2}{3}}}$$

Saint-Venant : not dispersive

Linear theory of Airy (see lecture of Julien Chauchat)

$$\omega^2 = gk \tanh(kh_0)$$

$$kh_0 \ll 1 \quad \omega^2 = gh_0 k^2 \left(1 - \frac{k^2 h_0^2}{3} \right) + gk O(k^5 h_0^5)$$

Serre-Green-Naghdi : weakly dispersive

Hyperbolicity

Hyperbolic system of equations

Mathematically

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{V}}{\partial x} = 0$$

matrix form, 1D case

Ex. Saint-Venant : $\mathbf{V} \begin{vmatrix} h \\ U \end{vmatrix}$

The system is hyperbolic if the eigenvectors of \mathbf{A} form a basis.

In particular, it is the case if the eigenvalues are real and distinct.

Exemple : Equations of Saint-Venant

$$\begin{aligned} \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} &= 0 \\ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial h}{\partial x} &= 0 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} U & h \\ g & U \end{bmatrix}$$

$$\begin{vmatrix} U - \lambda & h \\ g & U - \lambda \end{vmatrix} = 0$$

$$\lambda = U \pm \sqrt{gh}$$

Saint-Venant : hyperbolic

Eigenvalues : characteristic velocities

The Serre-Green-Naghdi equations are not hyperbolic.

Physically

Hyperbolic : no propagation at an infinite velocity.



All equations of physics should be hyperbolic !

Peshkov & Romenski 2016

Peshkov et al. 2018

Peshkov et al. 2020

Incompressibility : a pressure change at any point in a confined incompressible fluid is transmitted instantaneously throughout the fluid.

Numerical treatment of the non-localities

Numerically

Hyperbolic \longrightarrow local

The evolution of the variables in a cell depends only on the state in the **neighbouring cells**.

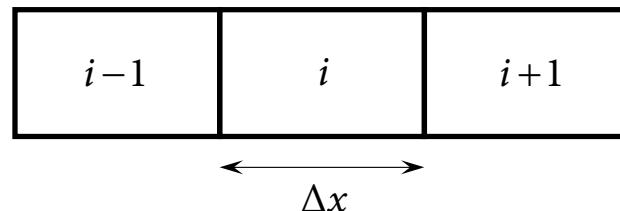
Time step

$$\Delta t = CFL \frac{\Delta x}{c_{\max}}$$

Courant-Friedrichs-Levy condition
Courant number < 1

Finite-volume method (Godunov-type)

Maximum characteristic velocity



Non-locality

\longrightarrow The evolution of the variables in a cell depends on **all cells of the domain** and of **all boundary conditions**.

\longrightarrow **Global system** to solve at each time step with the boundary conditions

\longrightarrow High computational time

Difficulty to treat the boundary conditions

especially with unstructured mesh

Serre-Green-Naghdi : “elliptic” step

Hyperbolization of the Serre-Green-Naghdi equations

Compressible fluids : pressure changes propagate at the sound velocity. $a = 1500 \text{ m} \cdot \text{s}^{-1}$ in water

However compressibility entails also static compressibility correction. $\rho = \rho(z)$

Hyperbolic approximations of the Serre-Green-Naghdi equations

Favrie & Gavrilyuk (2017)

Mathematical justification : Duchêne (2019)

Escalante *et al.* (2019)

Escalante & Morales de Luna (2020)

Bassi *et al.* (2020)

Richard (2021)

$$P = \frac{1}{h} \int_0^h \frac{p_N}{\rho} dz \quad \text{Depth-averaged non-hydrostatic pressure}$$

$$\frac{\partial h}{\partial t} + \frac{\partial h U}{\partial x} = 0$$

$$\frac{\partial h U}{\partial t} + \frac{\partial}{\partial x} \left(h U^2 + \frac{gh^2}{2} + h P \right) = 0$$

$$\varepsilon^2 \left(\frac{\partial w'}{\partial t'} + \frac{\partial u' w'}{\partial x'} + \frac{\partial w'^2}{\partial z'} \right) = -\frac{1}{F^2} \left(1 + \frac{\partial p'}{\partial z'} \right)$$

$$p_N(h) = 0$$

$$W = \langle w \rangle \\ \langle uw \rangle = UW + O(\varepsilon^2)$$

$$p'_N = \frac{h'^2 - z'^2}{2} \frac{\ddot{h}}{h}$$

$$p'_N(0) = \frac{h' \ddot{h}'}{2}$$

$$P' = \frac{h' \ddot{h}'}{3}$$

$$p'_N(0) = \frac{3}{2} P'$$

$$\frac{\partial w'}{\partial t'} + \frac{\partial u'w'}{\partial x'} + \frac{\partial w'^2}{\partial z'} = -\frac{1}{F^2} \frac{\partial p'_N}{\partial z'}$$

$$\int_0^{h'} \left(\frac{\partial w'}{\partial t'} + \frac{\partial u'w'}{\partial x'} + \frac{\partial w'^2}{\partial z'} \right) dz' = -\frac{1}{F^2} \int_0^{h'} \frac{\partial p'_N}{\partial z'} dz'$$

$$\frac{\partial h'W'}{\partial t'} + \frac{\partial h'U'W'}{\partial x'} = \frac{1}{F^2} p'_N(0) + O(\varepsilon^2)$$

$$\frac{\partial h'W'}{\partial t'} + \frac{\partial h'U'W'}{\partial x'} = \frac{1}{F^2} \frac{3}{2} P' + O(\varepsilon^2)$$

$$\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2} P$$

Acoustic Energy

$$w' = -z' \frac{\partial U'}{\partial x'} + O(\varepsilon^2)$$

$$W' = -\frac{h'}{2} \frac{\partial U'}{\partial x'} + O(\varepsilon^2)$$

$$2W + h \frac{\partial U}{\partial x} = 0 \quad \longrightarrow \quad \text{Serre-Green-Naghdi (not hyperbolic)}$$

$$2W' + h' \frac{\partial U'}{\partial x'} = -\varepsilon^2 M^2 \left(\frac{\partial h' P'}{\partial t'} + \frac{\partial h' U' P'}{\partial x'} \right)$$

Nombre de Mach $M = \frac{\sqrt{gh}}{a} = O(\varepsilon^\gamma), \quad \gamma > 0$

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \frac{gh^2}{2} + hP \right) = 0$$

$$\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2} P$$

$$\frac{\partial hP}{\partial t} + \frac{\partial hUP}{\partial x} = -a^2 \left(2W + h \frac{\partial U}{\partial x} \right)$$

Escalante *et al.* (2019)

This system admits an **exact** equation of **energy conservation**

$$\frac{\partial he}{\partial t} + \frac{\partial}{\partial x} (hUe + \Pi U) = 0$$

$$e = \frac{U^2}{2} + \frac{2}{3} W^2 + \frac{gh}{2} + \frac{P^2}{2a^2}$$

$$\Pi = \frac{gh^2}{2} + hP$$

acoustic energy

Hyperbolicity

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{V}}{\partial x} = \mathbf{S}$$

$$\mathbf{A} = \begin{bmatrix} U & h & 0 & 0 \\ g + \frac{P}{h} & U & 0 & 1 \\ 0 & 0 & U & 0 \\ 0 & a^2 & 0 & U \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} h \\ U \\ W \\ P \end{bmatrix}$$

$$\begin{cases} \lambda_{1,2} = U \\ \lambda_{3,4} = U \pm \sqrt{gh + P + a^2} \end{cases}$$

$a^2 \gg |P|$ 4 linearly independant eigenvectors

The system is **hyperbolic**.

In practice, a can be chosen much smaller than the physical value of the sound velocity

Artificially reduced sound velocity : Pseudo-compressible approach

Auclair *et al.* (2018)

$$\Delta t = CFL \frac{\Delta x}{c_{\max}}$$

$$c_{\max} \approx a$$

Increase the time step, reduce the computational time.

Dispersive properties equivalent to Serre-Green-Naghdi if $M < 0.1$

$$a < 100 \text{ m} \cdot \text{s}^{-1}$$

Faster calculation

Simpler implementation (ex. parallelization)

Boundary conditions simpler to implement

(Courtesy of Arnaud Duran)

Boundary Conditions (1)

Soliton between walls

Serre-Green-Naghdi equations

Generation with a relaxation method

Unstructured mesh

Problems at the walls !

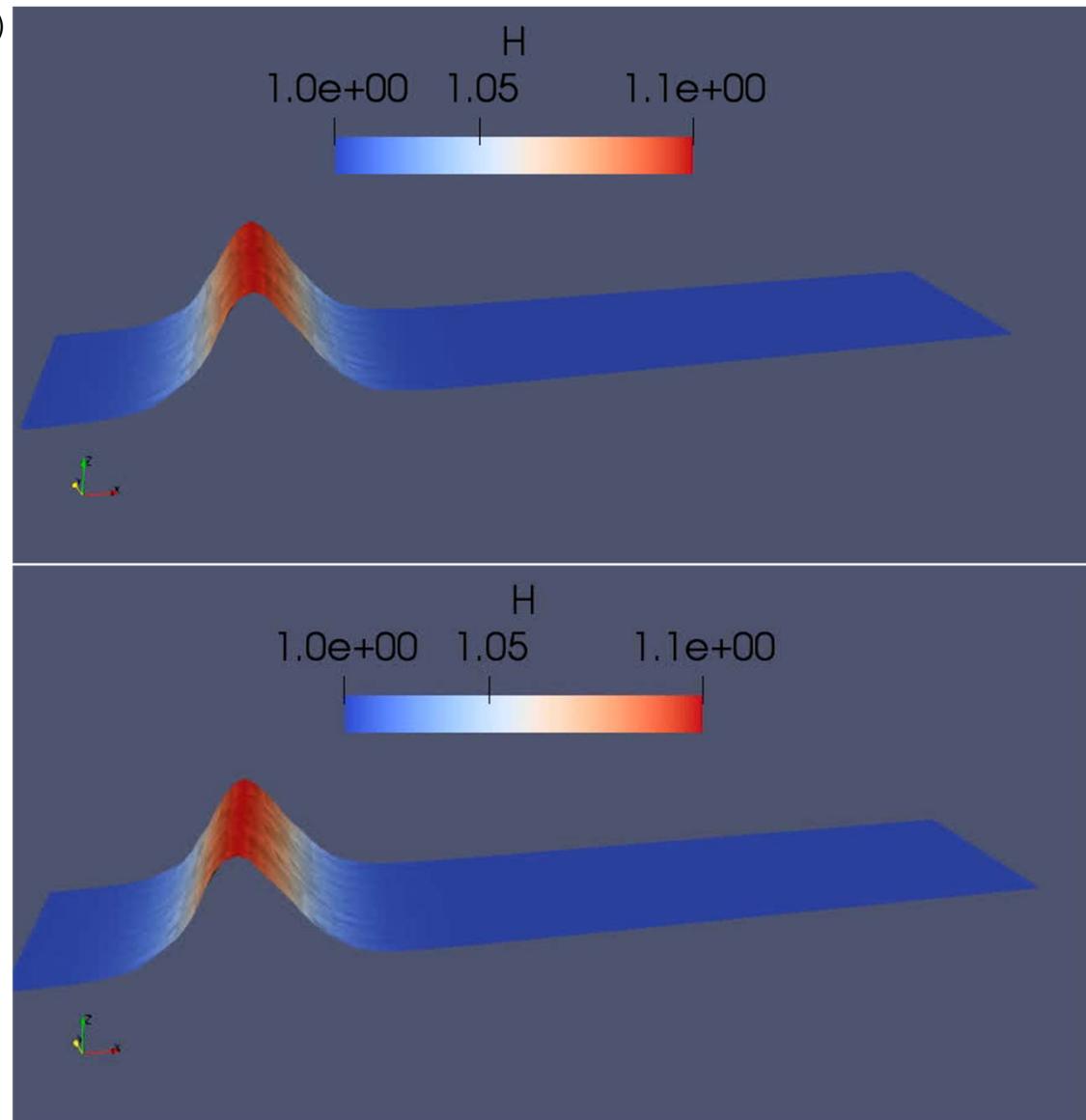
Can even cause the code to explode !

Periodic boundary conditions

Calibration of the relaxation parameters

The periodic boundary conditions are not always possible in operational context.

Complex with unstructured mesh



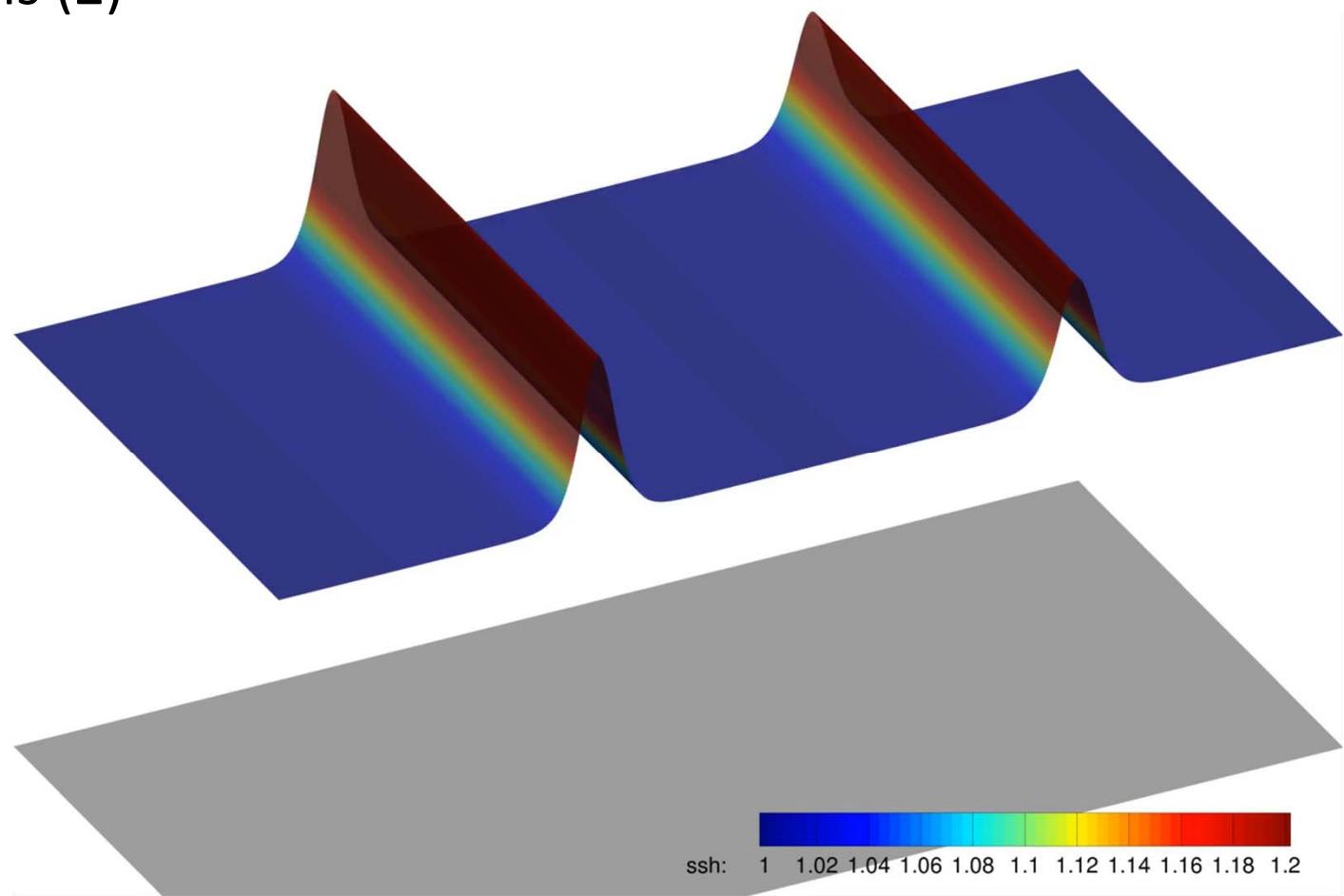
Boundary Conditions (2)

(courtesy of Frédéric Couderc)

2 Solitons between walls

Hyperbolic dispersive equations

No special treatment



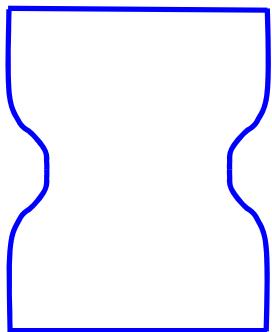
Boundary Conditions (3)

(courtesy of Frédéric Couderc)

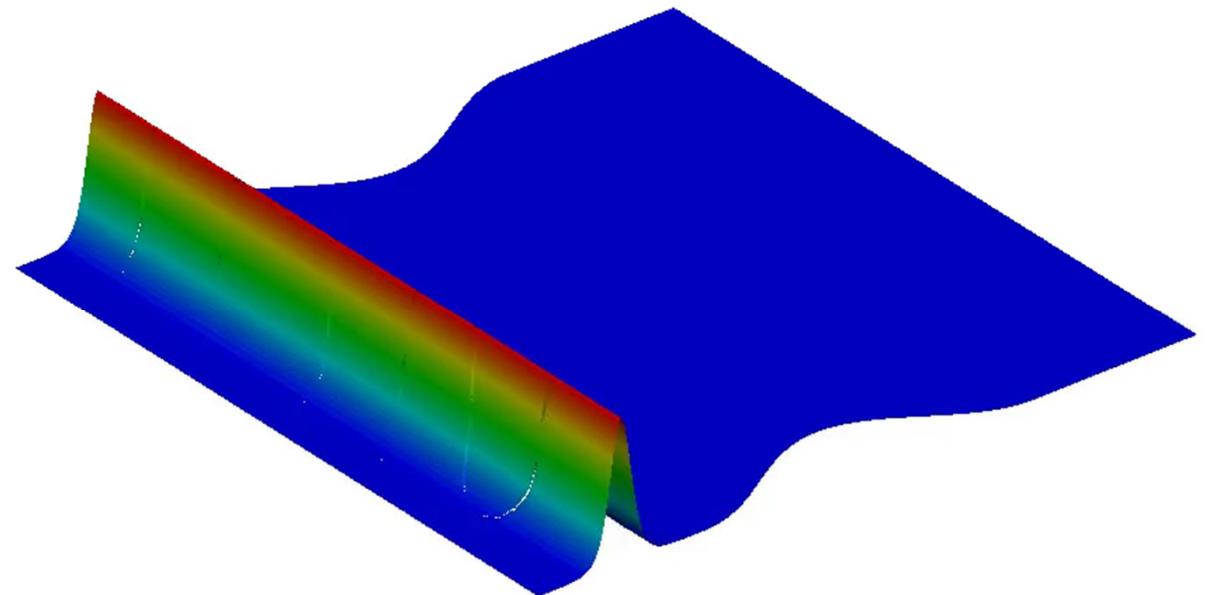
Soliton between walls with a lateral narrowing of the pool

Hyperbolic dispersive equations

Unstructured meshes



No special treatment



A **wall boundary condition** is easy to write with the hyperbolic equations !

Boundary Conditions (4)

(courtesy of Maria Kazakova)

Transparent boundary conditions

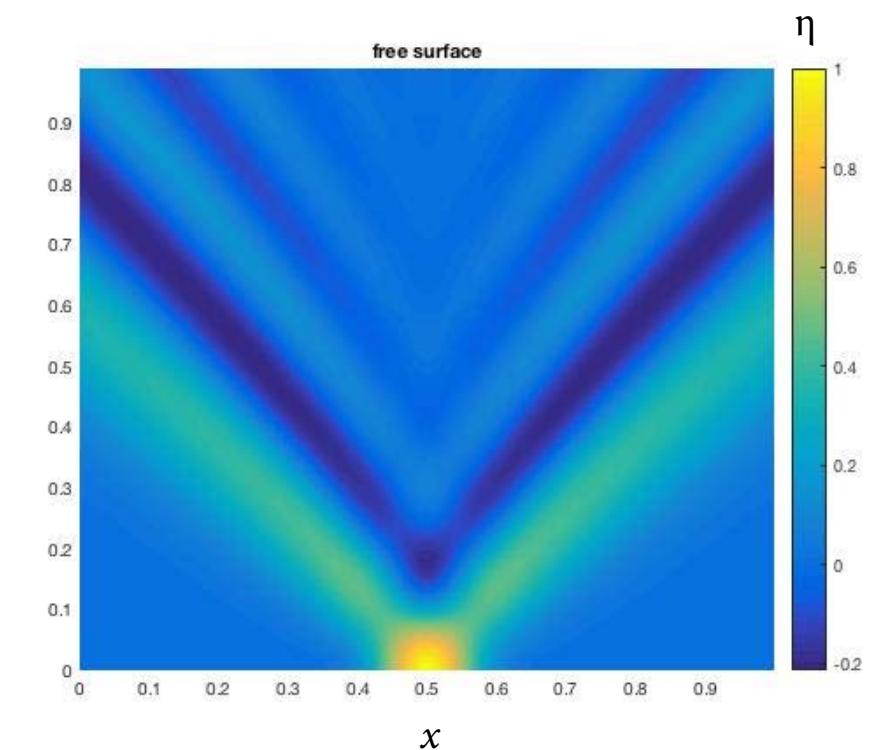
Linearized Serre-Green-Naghdi equations

Gaussian initial distribution for the free surface elevation

Zero initial velocity

Kazakova & Noble (2020)

The method is not applicable in the nonlinear case.



Boundary Conditions (5)

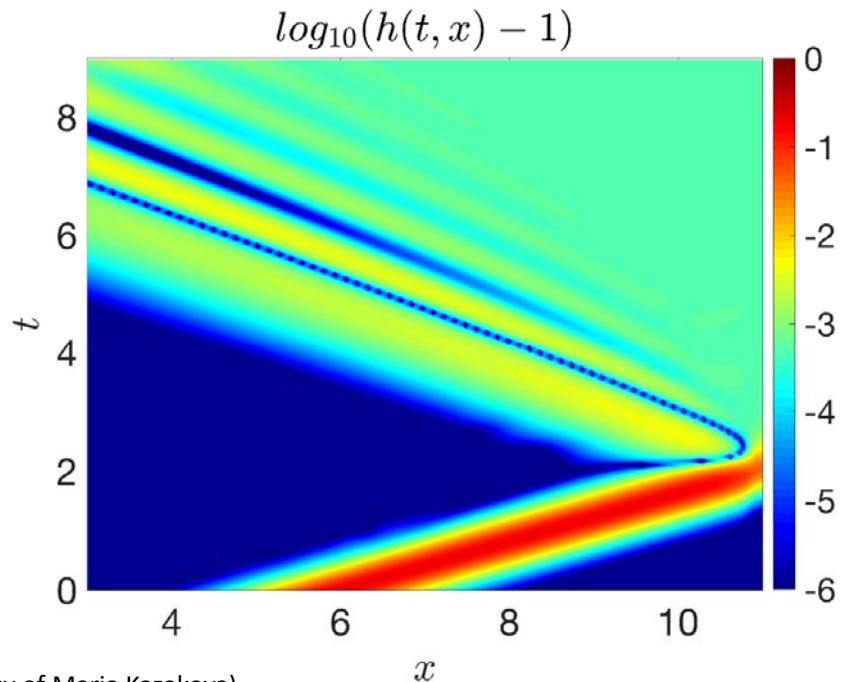
Propagation of a solitary wave

Nonlinearity : 0.5

Hyperbolic dispersive equations of Favrie & Gavrilyuk (2017)

Neumann boundary conditions

Reflection : 24 %



(courtesy of Maria Kazakova)

Method : Perfectly Matched Layers

Kazakova (2018)

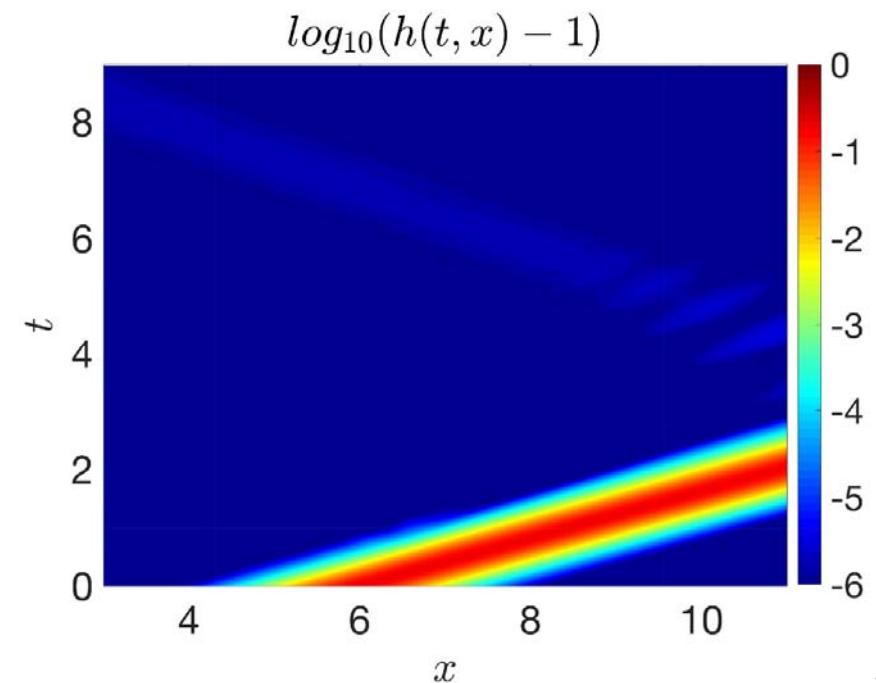
Easy to apply to hyperbolic systems.

Derived in the linear case.

Works well even in the **nonlinear case** (no mathematical proof)

Perfectly Matched Layers

Reflection : 0.8 %



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Numerical Scheme

Low Mach / Low Froude scheme for oceanic flow simulations

Simplified approach (1 layer, flat bottom, no source term)

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\mathbf{U}) = 0$$

$$\frac{\partial h\mathbf{U}}{\partial t} + \operatorname{div}(h\mathbf{U} \otimes \mathbf{U}) + h \operatorname{grad}(gh) = 0$$

Oceanic flows : low Froude number

$$F = \frac{U}{\sqrt{gh}} \ll 1$$

→ Classical Riemann solvers (Rusanov, HLL, HLLC...) are too much dissipative.

→ Better than a first-order HLLC scheme

First-order scheme equivalent or even better than a second-order HLLC scheme and much faster

Second-order extension (MUSCL, Heun) even better

Applicable to multi-layer shallow water model

Grenier, Vila & Villedieu (2013)

Parisot & Vila (2016)

Couderc, Duran & Vila (2017)

Explicit scheme

Centred + small stabilizing diffusive term related to the pressure gradient

Centred + another stabilizing term

Two stabilization constants to choose to ensure :

- Linear stability
- Strict decrease of the energy (nonlinear stability)
- While minimizing the dissipation

TOLOSA software
Couderc, Vila. et al



Shearing

$$u = U + u^*$$

$$v = V + v^*$$

By definition,

$$\langle u^* \rangle = 0$$

$$\langle v^* \rangle = 0$$

$$\langle u^2 \rangle = U^2 + \langle u^{*2} \rangle$$

$$\langle v^2 \rangle = V^2 + \langle v^{*2} \rangle$$

$$\langle uv \rangle = UV + \langle u^* v^* \rangle$$

→ Symmetrical **Tensor** of order 2

$$\Phi = \frac{1}{h^3} \int_0^{h'} (\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U}) dz$$

$$\Phi_{11} = \frac{\langle u^2 \rangle}{h^2} \quad \Phi_{12} = \frac{\langle uv \rangle}{h^2}$$

$$\langle u^2 \rangle = U^2 + h^2 \Phi_{11}$$

$$\langle v^2 \rangle = V^2 + h^2 \Phi_{22}$$

$$\langle uv \rangle = UV + h^2 \Phi_{12}$$

$$\Phi = \frac{1}{h^2} \langle (\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U}) \rangle$$

$$\Phi_{22} = \frac{\langle v^2 \rangle}{h^2}$$

Tensor **enstrophy**

New variable



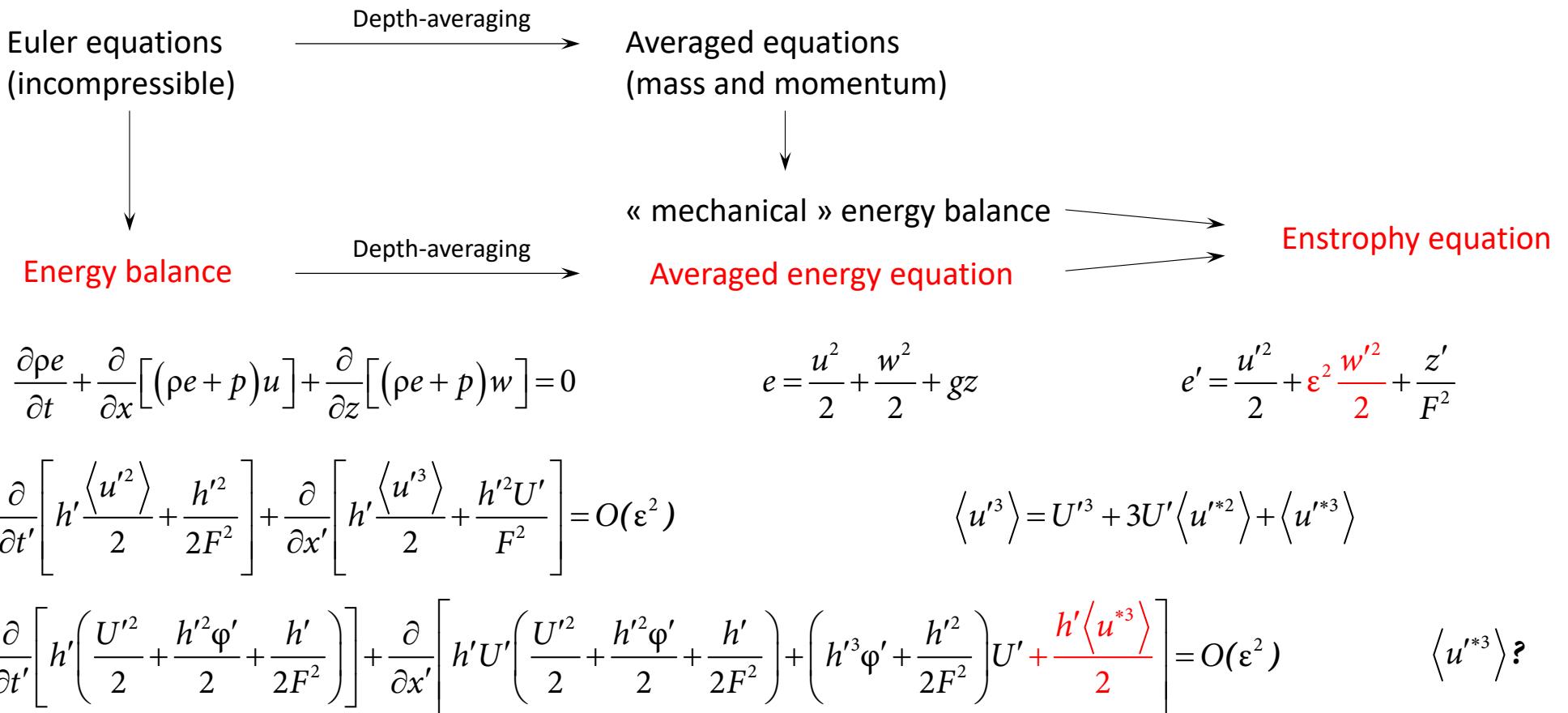
New equation

- Mass (scalar)
- Momentum (vectorial)
- Enstrophy (tensor)

→ Three-equation model

Enstrophy equation

1D case (2D flows)



Weakly sheared flows

Teshukov (2007)

$$u' = U' + \varepsilon^\beta u'^*$$

$$0 < \beta < 1$$

$$\langle u'^2 \rangle = U'^2 + \varepsilon^{2\beta} \langle u'^{*2} \rangle$$

$$\langle u'^3 \rangle = U'^3 + 3\varepsilon^{2\beta} U' \langle u'^{*2} \rangle + \varepsilon^{3\beta} \langle u'^{*3} \rangle$$

$$\frac{\partial}{\partial t'} \left[h' \left(\frac{U'^2}{2} + \varepsilon^{2\beta} \frac{h'^2 \varphi'}{2} + \frac{h'}{2F^2} \right) \right] + \frac{\partial}{\partial x'} \left[h' U' \left(\frac{U'^2}{2} + \varepsilon^{2\beta} \frac{h'^2 \varphi'}{2} + \frac{h'}{2F^2} \right) + \left(\varepsilon^{2\beta} h'^3 \varphi' + \frac{h'^2}{2F^2} \right) U' \right] = O(\varepsilon^{3\beta}) + O(\varepsilon^2)$$

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial h U}{\partial x} &= 0 \\ \frac{\partial h U}{\partial t} + \frac{\partial}{\partial x} (h U^2 + \Pi) &= 0 \\ \frac{\partial h e}{\partial t} + \frac{\partial}{\partial x} (h U e + \Pi U) &= 0 \end{aligned}$$

$$\begin{aligned} e &= \frac{U^2}{2} + \frac{h^2 \varphi}{2} + \frac{gh}{2} \\ \Pi &= h^3 \varphi + \frac{gh^2}{2} \end{aligned}$$

Structure of the Euler equations of compressible fluids (1D)

«density »	h
« pressure »	Π
« temperature »	$\langle u'^{*2} \rangle = h^2 \varphi$
« internal energy »	$\frac{h^2 \varphi}{2}$
« equation of state »	$\Pi = h \langle u'^{*2} \rangle = h^3 \varphi$
« entropy »	φ

Hyperbolicity

$$\left. \begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} &= 0 \\ \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \Pi \right) &= 0 \end{aligned} \right\} \longrightarrow \frac{\partial}{\partial t} \left[h \left(\frac{U^2}{2} + \frac{gh}{2} \right) \right] + \frac{\partial}{\partial x} \left[hU \left(\frac{U^2}{2} + \frac{gh}{2} \right) + \frac{gh^2}{2} U \right] + U \frac{\partial h^3 \varphi}{\partial x} = 0$$

$$\frac{\partial}{\partial t} \left[h \left(\frac{U^2}{2} + \frac{h^2 \varphi}{2} + \frac{gh}{2} \right) \right] + \frac{\partial}{\partial x} \left[hU \left(\frac{U^2}{2} + \frac{h^2 \varphi}{2} + \frac{gh}{2} \right) + \left(h^3 \varphi + \frac{gh^2}{2} \right) U \right] = 0 \longrightarrow \frac{\partial h \varphi}{\partial t} + \frac{\partial hU \varphi}{\partial x} = 0$$

$$\frac{\partial}{\partial t} \begin{bmatrix} h \\ U \\ \varphi \end{bmatrix} + \mathbf{A} \frac{\partial}{\partial x} \begin{bmatrix} h \\ U \\ \varphi \end{bmatrix} = 0$$

$$\mathbf{A} = \begin{bmatrix} U & h & 0 \\ g + 3h\varphi & U & h^2 \\ 0 & 0 & U \end{bmatrix}$$

$$\begin{aligned} \lambda_1 &= U \\ \lambda_{2,3} &= U \pm \sqrt{gh + 3h^2 \varphi} \end{aligned}$$

Hyperbolic system



Discontinuities in finite time

Conservation of :

- Mass
- Momentum
- Energy

Creation of enstrophy

The enstrophy plays the role of the entropy. **+ Dissipation**



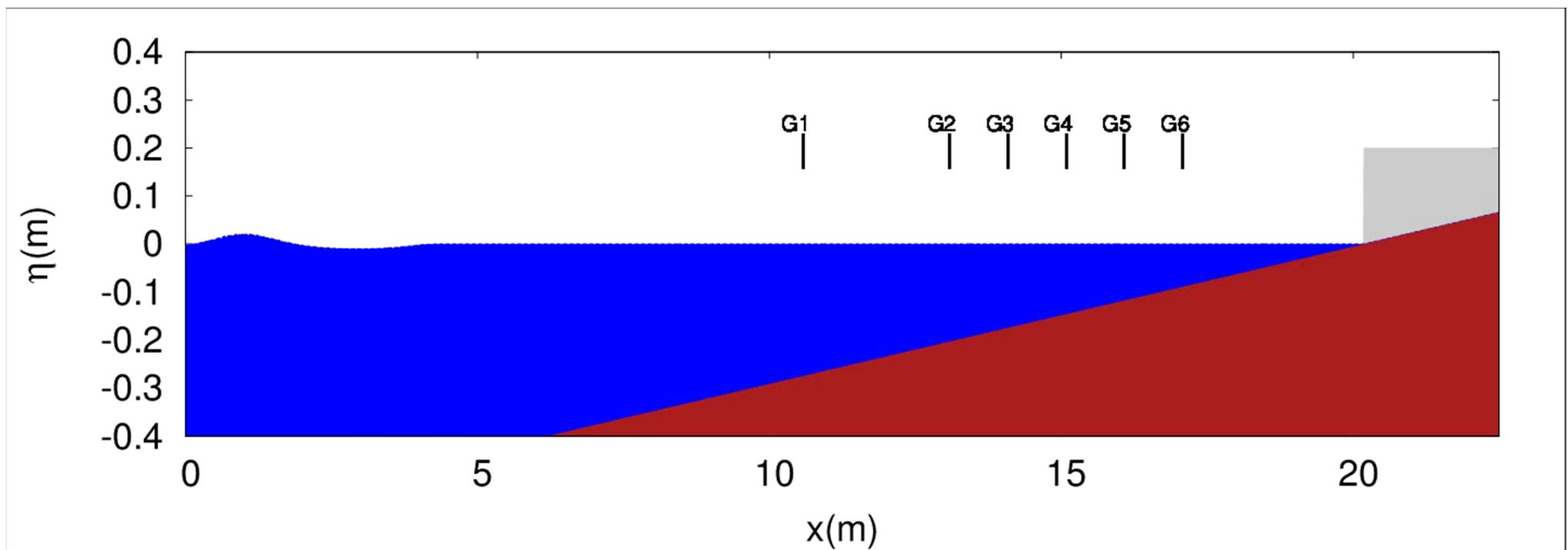
Breaking waves
Hydraulic jumps
Roll waves

Breaking waves (1)

- Introducing dissipation :
- Eddy viscosity
 - Roller models
 - Switching

Numerical perturbations can appear with switching

(courtesy of Arnaud Duran)



Breaking waves (2)

Averaging the LES equations

Large-scale turbulence : resolved → Enstrophy
 Small-scale turbulence : modelled → Eddy viscosity

(Mild slope case)

$$\begin{aligned}\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} &= 0 \\ \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \frac{gh^2}{2} + h^3\varphi + \frac{h^2\ddot{h}}{3} \right) &= \frac{\partial}{\partial x} \left(\frac{4}{R} h^3 \sqrt{\varphi} \frac{\partial U}{\partial x} \right) - gh \frac{db}{dx} \\ \frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} &= \frac{8h\sqrt{\varphi}}{R} \left(\frac{\partial U}{\partial x} \right)^2 - C_r h \varphi^{3/2}\end{aligned}$$

Energy

$$\begin{aligned}\frac{\partial he}{\partial t} + \frac{\partial}{\partial x} [hUe + \Pi U] &= \frac{\partial}{\partial x} \left(\frac{4}{R} h^3 U \sqrt{\varphi} \frac{\partial U}{\partial x} \right) - \frac{C_r}{2} h^3 \varphi^{3/2} \\ e &= \frac{U^2}{2} + \frac{gh}{2} + gb + \frac{h^2\varphi}{2} + \frac{\dot{h}^2}{6} \\ \Pi &= \frac{gh^2}{2} + h^3\varphi + \frac{h^2\ddot{h}}{3}\end{aligned}$$

→ Better description of breaking waves (front wave profile, asymmetry...)

Kazakova & Richard (2019)

Breaking waves (3)

Extension to the 2D case

Wave breaking over a reef and a beach

Simulation of the experiments of Swigler (2009)

Models of wave breaking with enstrophy :

Less sensitive to the breaking criterion

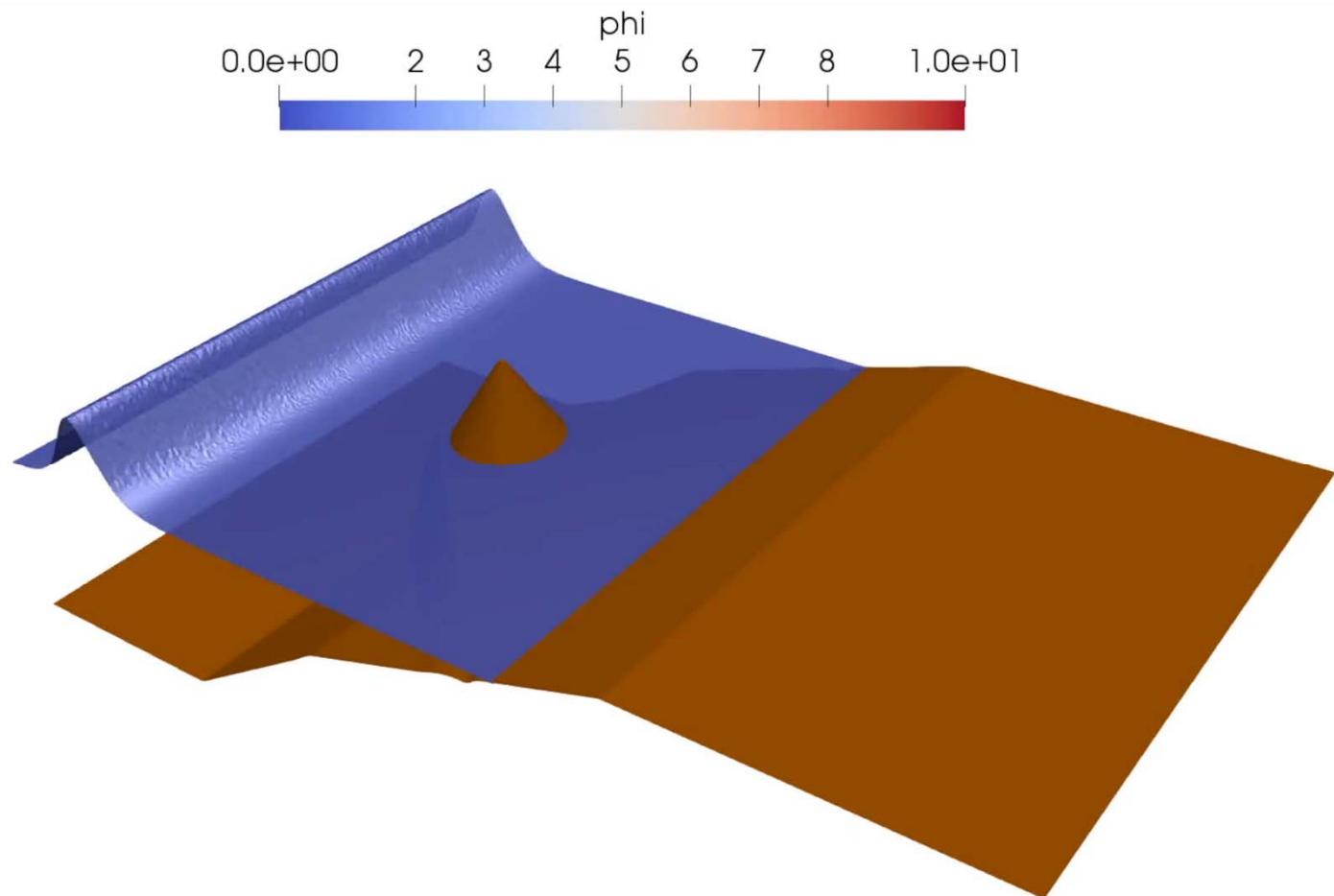
or no breaking criterion at all

Gavrilyuk, Liapidevskii & Chesnokov (2016)

Kazakova & Richard (2019)

Richard, Duran & Fabrèges (2019)

Duran & Richard (2020)



Hyperbolic dispersive + breaking ?

3D Reconstruction and friction

→ Asymptotic method

Dissipation

Constitutive law

Exemple : Newtonian fluids

$$\tau = 2\rho\nu \mathbf{D}$$

$$D = \frac{1}{2} \left[\mathbf{grad} \nu + (\mathbf{grad} \nu)^T \right]$$

Open-channel hydraulics

$$Re \gg 1 \longrightarrow$$

Model of turbulence

Mixing length
Eddy viscosity

$$\nu_T$$

$$\tau = 2\rho(\nu + \nu_T) \mathbf{D}$$

$$\nu_T = \sqrt{2} L_m^2 \sqrt{\mathbf{D} : \mathbf{D}}$$

$$L_m = \kappa z \left(1 - e^{-z^+ / A^+} \right) \sqrt{1 - \frac{z}{h}}$$

$$\nu_{eff} = \nu + \nu_T$$

Constant of Von Kármán

$$\kappa \approx 0,41$$

Turbulence due to bottom shearing

$$A^+ \approx 26$$

Not breaking

$$z^+ = z \frac{u_b}{\nu}$$

$$u_b = \sqrt{\frac{\tau_b}{\rho}}$$

Shallow- water scaling

$$\varepsilon = \frac{h_0}{L} \ll 1 \quad x' = \frac{x}{L} \quad y' = \frac{y}{L} \quad z' = \frac{z}{L} \quad h' = \frac{h}{h_0} \quad u' = \frac{u}{u_0} \quad v' = \frac{v}{u_0} \quad w' = \frac{w}{\varepsilon u_0} \quad t' = t \frac{u_0}{L} \quad p' = \frac{p}{\rho g h_0}$$

$$\tau = 2\rho(v + v_T)D$$

$$v_T = \sqrt{2L_m^2 \sqrt{D:D}}$$

$$L_m = \kappa z \left(1 - e^{-z^+/A^+}\right) \sqrt{1 - \frac{z}{h}}$$

$$L'_m = \frac{L_m}{\kappa h_0} \quad v'_T = \frac{v_T}{\kappa^2 h_0 u_0}$$

$$\tau'_{xz} = \frac{\tau_{xz}}{\rho \kappa^2 u_0^2} \quad \tau'_{yz} = \frac{\tau_{yz}}{\rho \kappa^2 u_0^2}$$

$$\tau'_{xx} = \frac{\tau_{xx}}{\varepsilon \rho \kappa^2 u_0^2} \quad \tau'_{yy} = \frac{\tau_{yy}}{\varepsilon \rho \kappa^2 u_0^2} \quad \tau'_{zz} = \frac{\tau_{zz}}{\varepsilon \rho \kappa^2 u_0^2} \quad \tau'_{xy} = \frac{\tau_{xy}}{\varepsilon \rho \kappa^2 u_0^2}$$

$$D = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \frac{\partial v}{\partial y} & \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

Dimensionless numbers :

$$Re = \frac{h_0 u_0}{\nu}$$

$$Re_{ML} = \frac{h_0 u_0}{\nu_e} = \frac{1}{\kappa^2}$$

$$\eta = \frac{Re_{ML}}{Re} = \frac{1}{\kappa^2 Re} = \frac{\nu}{\nu_e}$$

$$\eta = \varepsilon^{2+m} \quad m > 0$$

$$F = \frac{u_0}{\sqrt{gh_0}}$$

Dimensionless equations

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0$$

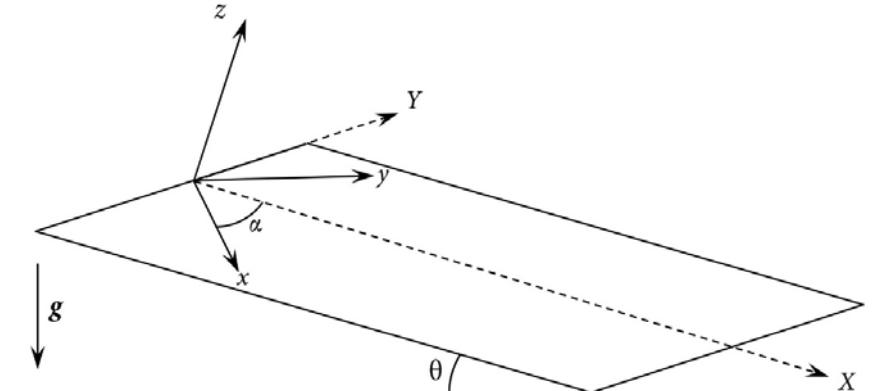
$$\left\{ \begin{array}{l} \frac{\varepsilon}{\kappa^2} \left(\frac{\partial u'}{\partial t'} + \frac{\partial u'^2}{\partial x'} + \frac{\partial u'v'}{\partial y'} + \frac{\partial u'w'}{\partial z'} \right) = \lambda \cos \alpha + \frac{\partial \tau'_{xz}}{\partial z'} - \frac{\varepsilon}{\kappa^2 F^2} \frac{\partial p'}{\partial x'} + O(\varepsilon^2) \\ \frac{\varepsilon}{\kappa^2} \left(\frac{\partial v'}{\partial t'} + \frac{\partial u'v'}{\partial x'} + \frac{\partial v'^2}{\partial y'} + \frac{\partial v'w'}{\partial z'} \right) = \lambda \sin \alpha + \frac{\partial \tau'_{yz}}{\partial z'} - \frac{\varepsilon}{\kappa^2 F^2} \frac{\partial p'}{\partial y'} + O(\varepsilon^2) \\ \frac{\partial p'}{\partial z'} = -\cos \theta + O(\varepsilon) \end{array} \right.$$

Hydrostatic pressure

$$\lambda = \frac{\sin \theta}{\kappa^2 F^2}$$

Boundary conditions

$$\begin{aligned} p'(h') &= O(\varepsilon) \\ \tau'_{xz}(h') &= O(\varepsilon^2) \\ \tau'_{yz}(h') &= O(\varepsilon^2) \end{aligned}$$



$$\nu'_{eff} = z'^2 (1-s) \sqrt{\left(\frac{\partial u'}{\partial z'} \right)^2 + \left(\frac{\partial v'}{\partial z'} \right)^2} + O(\varepsilon^2)$$

The molecular viscosity is negligible.

Integration over the depth

Mass $\frac{\partial h}{\partial t} + \operatorname{div}(h \mathbf{U}) = 0$

Momentum $\frac{\partial h' \mathbf{U}'}{\partial t'} + \operatorname{div}(h' \langle \mathbf{u}' \otimes \mathbf{u}' \rangle) + \operatorname{grad} \left(\frac{h'^2}{2F^2} \cos \theta \right) = \underbrace{\frac{\kappa^2}{\varepsilon} [h' \lambda - \tau'_{sh}(0)]}_{\text{Bottom friction}} + O(\varepsilon)$ $\lambda = \lambda \begin{vmatrix} \cos \alpha \\ \sin \alpha \end{vmatrix}$

Tensor enstrophy $\langle \mathbf{u}' \otimes \mathbf{u}' \rangle = \mathbf{U}' \otimes \mathbf{U}' + h'^2 \varphi'$ $\varphi' = \varphi \frac{h_0^2}{u_0^2}$

$$\frac{\partial h' \mathbf{U}'}{\partial t'} + \operatorname{div}(h' \mathbf{U}' \otimes \mathbf{U}' + h'^3 \varphi') + \operatorname{grad} \left(\frac{h'^2}{2F^2} \cos \theta \right) = \underbrace{\frac{\kappa^2}{\varepsilon} [h' \lambda - \tau'_{sh}(0)]}_{\text{relaxation}} + O(\varepsilon)$$

Equation of enstrophy

$$\frac{\partial h' \varphi'}{\partial t'} + \operatorname{div}(h' \varphi' \otimes \mathbf{U}') - 2h' \varphi' \operatorname{div} \mathbf{U}' + \operatorname{grad} \mathbf{U}' \cdot h' \varphi' + h' \varphi' \cdot (\operatorname{grad} \mathbf{U}')^\top = \frac{\kappa^2}{\varepsilon} \frac{1}{h'^2} [\mathbf{U}' \otimes \tau'_{sh}(0) + \tau'_{sh}(0) \otimes \mathbf{U}' - 2\mathbf{W}] + O(\mu^3) + O(\varepsilon)$$

$$W = \int_0^{h'} v'_{eff} \frac{\partial \mathbf{u}'}{\partial z'} \otimes \frac{\partial \mathbf{u}'}{\partial z'} dz'$$

Tensor of dissipation

$$\langle \mathbf{u}'^* \otimes \mathbf{u}'^* \otimes \mathbf{u}'^* \rangle$$

Asymptotic expansions

$$X = X_0 + \varepsilon X_1 + O(\varepsilon^2)$$

$$\begin{aligned} u &= u_0 + \varepsilon u_1 + O(\varepsilon^2) \\ \tau_{xz} &= \tau_{xz}^{(0)} + \varepsilon \tau_{xz}^{(1)} + O(\varepsilon^2) \end{aligned}$$

Momentum balance / Ox

$$\longrightarrow \tau'_{xz}^{(0)} \quad \tau'_{yz}^{(0)}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0$$

Constitutive law $\longrightarrow u'_0 \quad v'_0$

$$\begin{cases} \frac{\varepsilon}{\kappa^2} \left(\frac{\partial u'}{\partial t'} + \frac{\partial u'^2}{\partial x'} + \frac{\partial u'v'}{\partial y'} + \frac{\partial u'w'}{\partial z'} \right) = \lambda \cos \alpha + \frac{\partial \tau'_{xz}}{\partial z'} - \frac{\varepsilon}{\kappa^2 F^2} \frac{\partial p'}{\partial x'} + O(\varepsilon^2) \\ \frac{\varepsilon}{\kappa^2} \left(\frac{\partial v'}{\partial t'} + \frac{\partial u'v'}{\partial x'} + \frac{\partial v'^2}{\partial y'} + \frac{\partial v'w'}{\partial z'} \right) = \lambda \sin \alpha + \frac{\partial \tau'_{yz}}{\partial z'} - \frac{\varepsilon}{\kappa^2 F^2} \frac{\partial p'}{\partial y'} + O(\varepsilon^2) \\ \frac{\partial p'}{\partial z'} = -\cos \theta + O(\varepsilon) \end{cases}$$

Mass conservation $\longrightarrow w'_0$

$$\longrightarrow p'_0$$

$$\tau'_{sh} = \nu'_{eff} \frac{\partial u'}{\partial z'} + O(\varepsilon^2)$$

Momentum balance / Oz

$$\longrightarrow \tau'_{xz}^{(1)} \quad \tau'_{yz}^{(1)}$$

$$\nu'_{eff} = z'^2 (1-s) \sqrt{\left(\frac{\partial u'}{\partial z'} \right)^2 + \left(\frac{\partial v'}{\partial z'} \right)^2} + O(\varepsilon^2)$$

Momentum balance / Ox

$$\longrightarrow U_0 \quad U_1 \quad \Phi_0 \quad \Phi_1$$

Constitutive law $\longrightarrow u'_1 \quad v'_1$

Order 0 and viscous scaling

$$\tau'_{sh}^{(0)} = \lambda h' (1-s) \quad s = \frac{z}{h}$$

$$\tau'_{sh}^{(0)}(0) = \lambda h' \longrightarrow \frac{\kappa^2}{\varepsilon} [h' \lambda - \tau'_{sh}(0)] = -\kappa^2 \tau'_{sh}^{(1)}(0)$$

$$u'_0 = u'_0(h') + \frac{\lambda}{\lambda} \sqrt{\lambda h'} \ln s \quad \text{Log law} \longrightarrow \text{2nd scaling for a thin layer close to the bottom with viscosity}$$

$$\tilde{z} = \frac{z'}{\eta}; \quad \tilde{w} = \frac{w'}{\eta}; \quad \tilde{h} = \frac{h'}{\eta}; \quad \tilde{L}_m = \frac{L'_m}{\eta}; \quad \tilde{\nu} = \frac{\nu'}{\eta}.$$

$$\tilde{L}_m = \tilde{z} \sqrt{1-s} \left[1 - \exp \left(-\frac{\tilde{z} \sqrt{\tilde{\tau}_b}}{\kappa A^+} \right) \right] \quad \tilde{\nu}_{eff} = 1 + \tilde{\nu}_T$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \quad \text{Mass}$$

The molecular viscosity is not negligible.

$$\frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} = O(\eta); \quad \frac{\partial \tilde{\tau}_{yz}}{\partial \tilde{z}} = O(\eta); \quad \frac{\partial \tilde{p}}{\partial \tilde{z}} = O(\eta) \quad \text{Momentum}$$

Matching

Asymptotic expansion in the viscous layer

Matching with the expansions in the shallow-water layer

$$\frac{\partial \tilde{\tau}_{sh}^{(0)}}{\partial \tilde{z}} = 0$$

$$\tilde{\tau}_{sh}^{(0)} = \text{constant}$$

Matching :

$$\tilde{\tau}_{sh}^{(0)} = \tau_{sh}'^{(0)}(\mathbf{0}) = h' \lambda$$

Asymptotic matching for the velocity in an overlap layer

$$z = \sqrt{\eta} b h$$

$$\tilde{u}_0 = \sqrt{\lambda h'} \frac{\lambda}{\lambda} \left[-\frac{\xi}{1 + \sqrt{1 + \xi^2}} + \ln \left(\xi + \sqrt{1 + \xi^2} \right) + \mathcal{R}(\xi) \right]$$

$$\xi = 2\sqrt{\lambda h'} \tilde{z}$$

$$A = 2\kappa A^+$$

$$\mathcal{R}(\xi) = \int_0^\xi \frac{d\xi}{1 + \sqrt{1 + \xi^2} \left(1 - e^{-\xi/A} \right)^2} - \int_0^\xi \frac{d\xi}{1 + \sqrt{1 + \xi^2}} + O(\sqrt{\eta})$$

$$R = \int_0^\infty \frac{d\xi}{1 + \sqrt{1 + \xi^2} \left(1 - e^{-\xi/A} \right)^2} - \int_0^\infty \frac{d\xi}{1 + \sqrt{1 + \xi^2}}$$

$$u'_0(s = \sqrt{\eta} b) = \tilde{u}_0 \left(\xi = \frac{2b\sqrt{\lambda h'^3}}{\sqrt{\eta}} \right) + O(\sqrt{\eta})$$

Order of magnitude of shearing

We obtain

$$u'_0(h) = \frac{\lambda}{\lambda} \sqrt{\lambda h'} (R - 1 + \ln 2 + \ln M - \ln \eta)$$

$$M = 2\sqrt{\lambda h'^3}$$

let

$$\mu = -\frac{2}{\ln \eta} \quad \text{small} \quad \varepsilon < \mu < 1$$

$$\mu = -\frac{2}{2+m} \frac{1}{\ln \varepsilon} \quad \forall p > 0, \lim_{\varepsilon \rightarrow 0} (\varepsilon \ln^p \varepsilon) = 0$$

If ε is small enough, $\varepsilon < \mu^p$ $p \in \mathbb{N}$

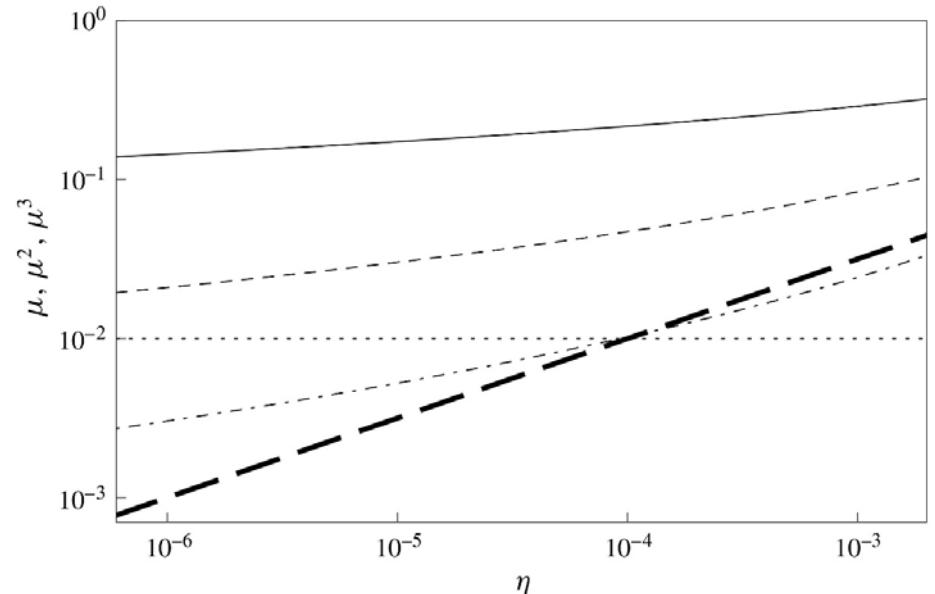
$$u'_0(h) = \frac{\lambda}{\lambda} \frac{\sqrt{\lambda h'}}{\mu} [2 + \mu(R - 1 + \ln 2 + \ln M)]$$

We impose $u'_0(h) = O(1)$ $\lambda = \mu^2 \lambda_0$, $\lambda_0 = O(1)$

$$u'_0(h) = \frac{\lambda}{\lambda} \sqrt{\lambda_0 h'} [2 + \mu(R - 1 + \ln 2 + \ln M)]$$

$$C(\mu) = 2 + \mu(R - 1 + \ln 2 + \ln M)$$

$$u'_0(h) = C(\mu) \frac{\lambda}{\lambda} \sqrt{\lambda_0 h'}$$



We obtain :

$$\varphi'_0 = O(\mu^2)$$

$$\langle u'^*_0 \otimes u'^*_0 \otimes u'^*_0 \rangle = O(\mu^3)$$

Weak shearing if μ is small.

Equations of Saint-Venant

$$\frac{\partial h' \mathbf{U}'}{\partial t'} + \mathbf{div}(h' \mathbf{U}' \otimes \mathbf{U}') + \mu^2 h'^3 \varphi' + \mathbf{grad} \left(\frac{h'^2}{2F^2} \cos \theta \right) = -\frac{\kappa^2}{\varepsilon} \tau'_{sh}^{(1)}(\mathbf{0}) + O(\varepsilon)$$

The terms $O(\mu^2)$ are neglected \longrightarrow Saint-Venant

With the asymptotic expansions, we can write :

$$\tau'_{sh}^{(1)}(\mathbf{0}) = \frac{\mu^2}{C^2(\mu)} \left(\|\mathbf{U}'_0\| \mathbf{U}'_1 + \mathbf{U}'_0 \frac{\mathbf{U}'_0 \cdot \mathbf{U}'_1}{\|\mathbf{U}'_0\|} \right) + O(\mu^2)$$

$$\|\mathbf{U}'_0\| \mathbf{U}'_1 + \mathbf{U}'_0 \frac{\mathbf{U}'_0 \cdot \mathbf{U}'_1}{\|\mathbf{U}'_0\|} = \frac{1}{\varepsilon} (\|\mathbf{U}'\| \mathbf{U}' - \|\mathbf{U}'_0\| \mathbf{U}'_0)$$

$$\|\mathbf{U}'_0\| \mathbf{U}'_0 = \frac{C^2(\mu)}{\mu^2} h' \boldsymbol{\lambda}$$

Let $C_f = \frac{\mu^2 \kappa^2}{C^2(\mu)}$ Friction Coefficient

$$-\kappa^2 \tau'_{sh}^{(1)}(\mathbf{0}) = \frac{1}{\varepsilon} (\kappa^2 h' \boldsymbol{\lambda} - C_f \|\mathbf{U}'\| \mathbf{U}') + O(\mu^2)$$

$$\frac{\partial h \mathbf{U}}{\partial t} + \mathbf{div}(h \mathbf{U} \otimes \mathbf{U}) + \mathbf{grad} \left(\frac{gh^2}{2} \cos \theta \right) = \mathbf{g}_T h - C_f \|\mathbf{U}\| \mathbf{U}$$

$$\mathbf{g}_T = g \sin \theta \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$$

Coefficient de frottement

Explicit expression : $\frac{\kappa}{\sqrt{C_f}} = \frac{2\kappa\sqrt{2}}{\sqrt{f}} = R - 2 + 2\ln 2 + \ln \kappa + \ln \frac{\sqrt{gh^3 \sin \theta}}{v}$ Darcy $f = 8C_f$

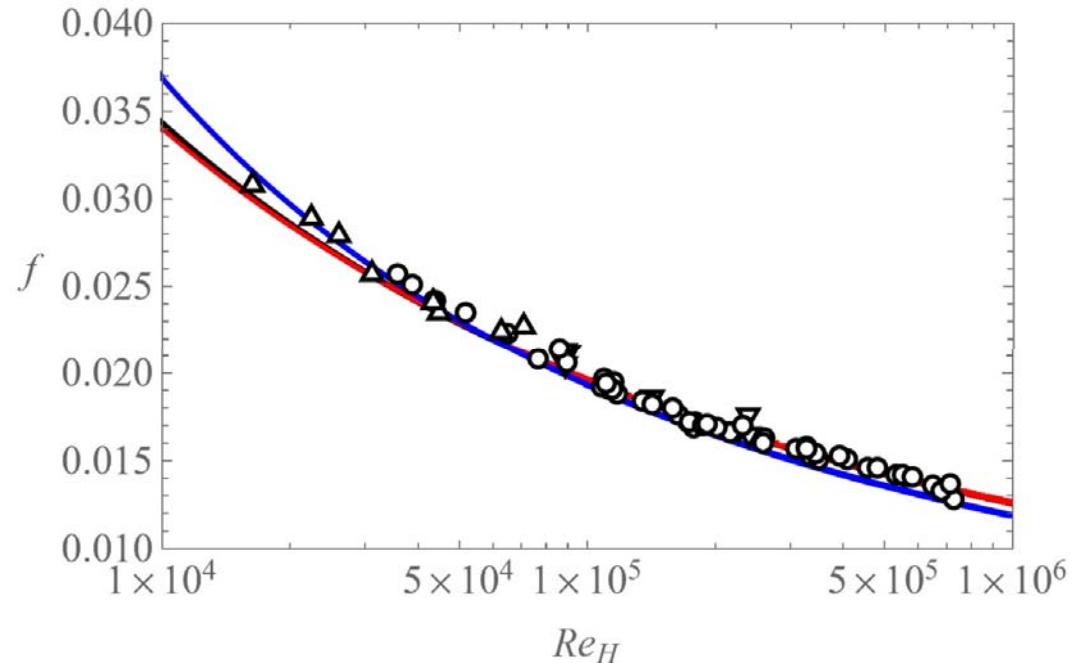
Valid in equilibrium or non-equilibrium

A more convenient approximate explicit expression :

$$C_f = \frac{\kappa^2}{\ln^2(\kappa^2 Re)} \quad Re = \frac{h\|U\|}{v}$$

More precise :

$$\frac{\kappa}{\sqrt{C_f}} = \frac{2\kappa\sqrt{2}}{\sqrt{f}} = R - 2 + 2\ln 2 + \ln \kappa + \ln \frac{\kappa h \|U\|}{v \ln(\kappa^2 Re)}$$



Equations with (weak) shearing

We keep terms $O(\mu^2)$ and we neglect terms $O(\mu^3)$

We obtain the system

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\mathbf{U}) = 0$$

$$\frac{\partial h\mathbf{U}}{\partial t} + \operatorname{div}(h\mathbf{U} \otimes \mathbf{U} + h^3 \boldsymbol{\varphi}) + \operatorname{grad}\left(\frac{gh^2}{2} \cos \theta\right)$$

$$= \left(1 - \frac{\alpha_1}{\kappa} \sqrt{C_f}\right) (\mathbf{g}_T h - C_f \mathbf{U} \|\mathbf{U}\|) + \alpha \left(\kappa - \alpha_1 \sqrt{C_f}\right) h \sqrt{C_f} \frac{\mathbf{g}_T}{g_T} \left(h \operatorname{tr} \boldsymbol{\varphi} - \frac{g_T}{\kappa^2}\right) - \kappa \alpha_1 h \sqrt{C_f} \left(h \boldsymbol{\varphi} \cdot \frac{\mathbf{g}_T}{g_T} - \frac{\mathbf{g}_T}{\kappa^2}\right)$$

$$\frac{\partial h\boldsymbol{\varphi}}{\partial t} + \operatorname{div}(h\boldsymbol{\varphi} \otimes \mathbf{U}) - 2h\boldsymbol{\varphi} \operatorname{div} \mathbf{U} + \operatorname{grad} \mathbf{U} \cdot h\boldsymbol{\varphi} + h\boldsymbol{\varphi} \cdot (\operatorname{grad} \mathbf{U})^T$$

$$= \frac{\alpha_2}{\kappa} \frac{\sqrt{C_f}}{h^2} \left[\mathbf{U} \otimes (C_f \mathbf{U} \|\mathbf{U}\| - \mathbf{g}_T h) + (C_f \mathbf{U} \|\mathbf{U}\| - \mathbf{g}_T h) \otimes \mathbf{U} \right] - \alpha \alpha_2 \frac{C_f}{h} \left(\mathbf{U} \otimes \frac{\mathbf{g}_T}{g_T} + \frac{\mathbf{g}_T}{g_T} \otimes \mathbf{U} \right) \left(h \operatorname{tr} \boldsymbol{\varphi} - \frac{g_T}{\kappa^2} \right)$$

$$- \kappa \alpha_2 \frac{\sqrt{C_f}}{h} \left[\mathbf{U} \otimes \left(h \boldsymbol{\varphi} \cdot \frac{\mathbf{g}_T}{g_T} - \frac{\mathbf{g}_T}{\kappa^2} \right) + \left(h \boldsymbol{\varphi} \cdot \frac{\mathbf{g}_T}{g_T} - \frac{\mathbf{g}_T}{\kappa^2} \right) \otimes \mathbf{U} \right]$$

$$R_1 = \int_0^{+\infty} \frac{d\xi}{\sqrt{1+\xi^2 \left(1-e^{-\xi/A}\right)^2}} - \int_0^{+\infty} \frac{d\xi}{\sqrt{1+\xi^2}}$$

$$\alpha = R_1 - R + 1$$

$$\alpha_2 = \frac{1}{2(\zeta(3)-1)}$$

$$\alpha_1 = \alpha - \alpha_2$$

$$\kappa = 0,41; \quad A^+ = 26$$

$$R = 2,67; \quad R_1 = 4,82;$$

$$\alpha = 3,15; \quad \alpha_1 = 0,680;$$

$$\alpha_2 = 2,47$$

Reconstruction of the 3D fields

We can reconstruct the 3D velocity field.

$$\begin{aligned} \mathbf{u} = & \mathbf{U} \left[1 + \frac{\sqrt{C_f}}{\kappa} (1 + \ln s) \right] - \alpha \frac{\sqrt{C_f}}{\kappa^2} \left(\sqrt{C_f} \mathbf{U} \cdot \frac{\mathbf{g}_T}{g_T} - \sqrt{gh \sin \theta} \right) \frac{\mathbf{g}_T}{g_T} (1 + \ln s) \\ & + \alpha_2 \left[1 - \frac{\pi^2}{6} + \text{Li}_2(1-s) \right] \left[\left(\frac{\sqrt{C_f}}{\kappa} \mathbf{U} \cdot \frac{\mathbf{g}_T}{g_T} - h \sqrt{\text{tr} \Phi} \right) \frac{\mathbf{g}_T}{g_T} - \alpha \frac{\sqrt{C_f}}{\kappa^2} \left(\sqrt{C_f} \mathbf{U} \cdot \frac{\mathbf{g}_T}{g_T} - \sqrt{gh \sin \theta} \right) \frac{\mathbf{g}_T}{g_T} \right] \end{aligned}$$

$$s = \frac{z}{h}$$

$$\text{Li}_2(s) = - \int_0^s \frac{\ln(1-t)}{t} dt$$

It is possible to calculate w or $\tau_{sh}(\mathbf{0})$

With the **asymptotic method** and with a **model of turbulence**,

We have obtained an **explicite** expression of **friction**, including outside the equilibrium,

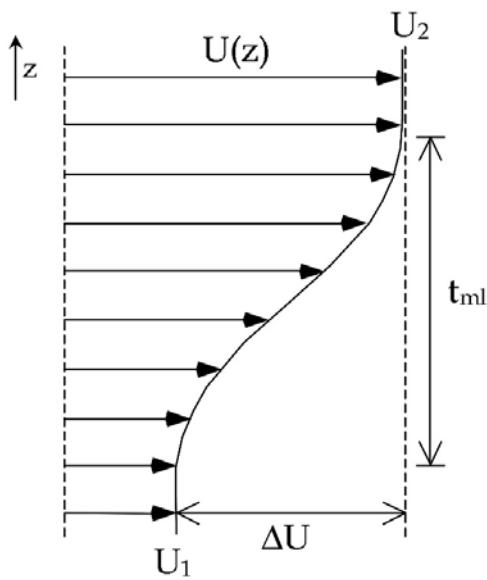
We have kept the information on the **3D fields**

And we have obtained corrective terms due to bottom **shearing**.

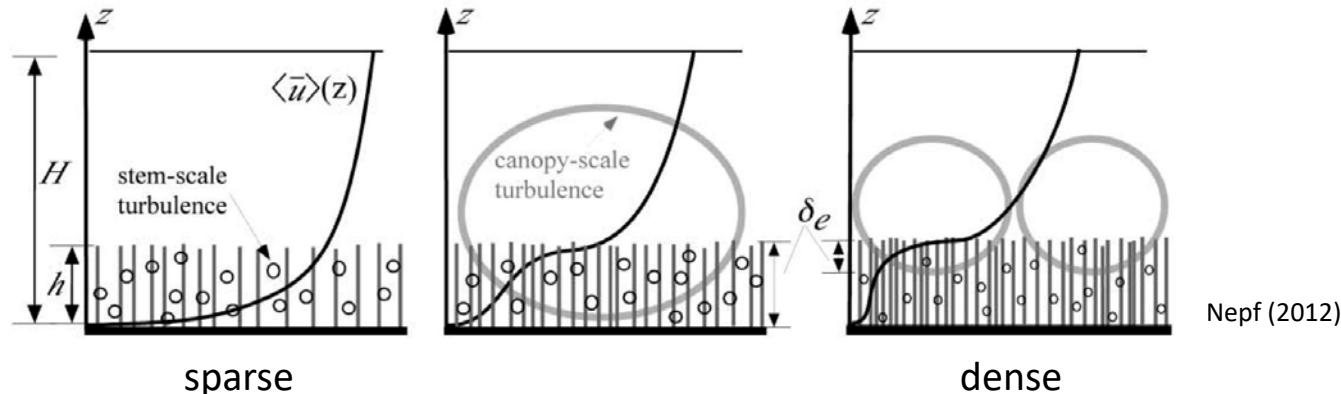
Open problems

Rough bottom

Vegetated flows



Ghisalberti & Nepf (2002)



Analogy with mixing layers

Kelvin-Helmholtz instability,
large vortices

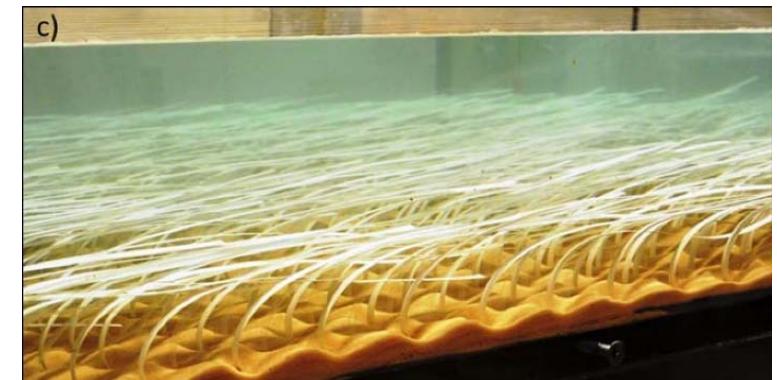
Turbulence

Important for sediment transport

Turbulence model ?

Asymptotic expansions ?

Model ?



Flexible vegetation

Experiments from Le Bouteiller & Venditti (2015)

Periodic waving of the vegetation related to turbulent structures
(monami)

